Relaxation effects on the flow over slender bodies

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The effects of heat-capacity lag on the flow over slender bodies are examined by means of an extension of Ward's (1949) generalized treatment of the slender-body problem. The results are valid for smooth bodies of arbitrary cross-sectional shape and attitude in the complete Mach number range up to, but not including, hypersonic conditions. Transonic flow can be treated owing to the presence of a dissipative mechanism in the basic differential equation, but the results in this Mach number range are probably of limited practical value.

The results show that cross-wind forces are unaffected to a first approximation, but that drag forces comparable with laminar skin-friction values can arise as a result of the relaxation of the internal degrees of freedom. The magnitude and sign of these effects depend strongly on body shape and free-stream Mach number.

Results are given for the surface pressure coefficient, and the variations of translational and internal mode temperature on and near the body are also found. The influence of these latter effects on heat transfer to the body is discussed.

1. Introduction

In the sections to follow we shall examine the flow of a polyatomic gas about slender three-dimensional bodies, taking into account the effects of heat-capacity lag. To simplify the problem we shall assume that the gas is inviscid and non-heat conducting, although we shall attempt, very briefly, to indicate how these gas properties would affect the simpler flow patterns obtained here.

From a practical point of view, relaxation effects are likely to assume greater significance as the general level of gas pressure decreases. The relaxation times for adjustment of internal molecular states to conditions of thermal equilibrium vary inversely with pressure, and it is conceivable that we may find 'relaxation lengths' comparable with over-all body length in some circumstances. ('Relaxation length' is defined as the product of relaxation time τ and free-stream speed U.) Provided that the number of collisions required to excite the internal mode is sufficiently large, it is possible to encounter such conditions within the régime of continuum flow. We treat the problem on the basis of this possibility.

The analysis to follow makes no explicit reference to any particular gas or mixture of gases, but some examples are given for CO_2, a gas which exhibits some of the required effects in circumstances which are practically realizable.

The effects of internal-mode relaxations on gas-dynamic behaviour have been investigated previously by Gunn (1952), who gave an account of sound absorption and dispersion, shock-wave effects, one-dimensional nozzle flows, and of the drag experienced by an object as a result of the dissipative actions of relaxation (the latter confined to two dimensions). The drag problems treated by Gunn are generally tackled by finding the perturbations introduced into a 'non-relaxing' gas flow, and he does not consider questions of transonic or supersonic flow. Here we attempt to 'unify' the treatment in the manner of Ward's (1949) solution of the inert-gas slender-body problem. The general question of relaxing and reacting gas flows has received a certain amount of attention recently, and it is pertinent to remark here on the formal similarity which exists between effects of heat-capacity lag and those which arise when endothermic chemical reactions occur in a gas flow. This similarity is made apparent by the fundamental work of Kirkwood & Wood (1957), and appears also in recent papers by Moore & Gibson (1960) and Vincenti (1960). An analysis of the supersonic flow of a chemically reacting gas round a sharp corner which would apply equally well to a relaxation effect problem has been given by Clarke (1960*a*). Moore & Gibson based their considerations on a velocity potential which satisfies the telegraph equation. The latter can be shown to approximate to the 'exact' small-perturbation equation for cases where the differences between frozen and equilibrium sound speeds are small. Vincenti considered the 'wavy-wall' problem, thereby dealing with a type of fundamental solution of the steady two-dimensional small-perturbation equation for reacting or relaxing gas flows, using, as does Clarke, the 'exact' small-disturbance equation. The one-dimensional unsteady analogue of Vincenti's problem (the harmonically oscillating piston) has been examined by Clarke (1958), and the step-input piston problem in a reacting gas has been treated by Chu (1957).

Most of the work to date, therefore, has been concerned with 'fundamental' types of solutions, or 'inputs', to the gas. One may perhaps level some criticism, from a practical point of view, at the 'wavy-wall' and harmonically oscillating piston solutions, since it is an essential feature of relaxation or reaction effects that entropy shall be produced continuously during the time taken to reach a new equilibrium state. The infinite past histories of these processes would therefore, strictly, require an infinite difference of entropy between the postulated 'undisturbed' regions and those adjacent to the object. Such (rather pedantic) objections are removed when the disturbing influence is of finite physical size (or duration).

As noted by Vincenti, the inclusion of a dissipative mechanism into the analysis permits one to obtain continuous solutions from a purely linear equation right through the 'transonic' region. Indeed, the presence of an infinity of sound speeds, ranging from the frozen to full equilibrium values, smears the transonic region over a finite band of free-stream velocities, so that the singular behaviour of the linear flow equations is confined to only one frequency of the infinite range which must be superimposed to summarize the effect of the obstacle as a whole.

2. The equations

We shall assume in what follows that only one internal energy mode exhibits significant relaxation effects. Any other modes are treated as 'active' modes, and it is supposed that their energy content is specified once the translational temperature T_1 is known. Following Kirkwood & Wood (1957), it is assumed that the state of the relaxing mode is described by another temperature T_2, which only equals T_1 when complete thermal equilibrium prevails. The specific internal energy e of the gas is made up as follows:

$$e = e_1 + e_2, \tag{1}$$

where

$$e_1 = \int_0^{T_1} C_{v1}\, dT_1, \tag{2}$$

$$e_2 = \int_0^{T_2} C_{v2}\, dT_2. \tag{3}$$

Here C_{v1} and C_{v2} are the specific heats at constant volume for the translational plus active degrees of freedom of the molecule and for the relaxing mode, respectively. When e_2 is in equilibrium with e_1, the upper limit of integration in equation (3) is replaced by T_1. We write C_v for the total specific heat at constant volume, i.e.

$$C_v = C_{v1} + C_{v2}. \tag{4}$$

(When C_{v1} and C_{v2} are temperature dependent, equation (4) only has a meaning when $T_1 = T_2$.) Note that a similar notation is followed by the specific heats at constant pressure. We shall write

$$C_p = C_{p1} + C_{v2}. \tag{5}$$

The pressure p, density ρ, and translational temperature are related by

$$p = \rho R T_1, \tag{6}$$

where R is the appropriate gas constant per unit mass. The gas is therefore treated as thermally perfect.

The relaxation of the internal mode to an equilibrium state is assumed to be described by the linear law

$$\tau' \frac{De_2(T_2)}{Dt} = e_2(T_1) - e_2(T_2), \tag{7}$$

where τ' is therefore the relaxation time, D/Dt the usual convective operator, and $e_2(T_1)$ signifies that the internal mode has an energy content appropriate to the actual, local, translational temperature. Since we are to deal with small perturbations from an originally undisturbed (equilibrium) stream, it is reasonable to assume that τ' is constant throughout. Furthermore, since T_2 will not differ greatly from T_1 in these circumstances, equation (3) shows that we can approximate equation (7) by

$$\tau' \frac{DT_2}{Dt} = T_1 - T_2. \tag{8}$$

This is tantamount to saying that C_{v2} is a constant, evaluated at the free-stream temperature T_∞; and, in the small-disturbance problem, we may say likewise for C_{v1}.

The energy equation can be written as

$$C_{v1} \frac{DT_1}{Dt} + C_{v2} \frac{DT_2}{Dt} + \frac{p}{\rho} \operatorname{div} \mathbf{q} = 0, \tag{9}$$

and C_{v1} and C_{v2} can subsequently be treated as constants. Here $\mathbf{q}$ is the gas velocity vector, and we do not introduce any further linearization at the present stage. DT_1/Dt can be eliminated from equation (9) by means of (8); and, using the mass conservation equation in the form derived by Kirkwood & Wood, it can be shown that

$$\frac{C_{v1}\tau'}{C_v}\frac{D}{Dt}\left\{\frac{1}{\rho}\frac{Dp}{Dt}+a_1^2\operatorname{div}\mathbf{q}\right\}+\frac{1}{\rho}\frac{Dp}{Dt}+a_2^2\operatorname{div}\mathbf{q} = 0. \tag{10}$$

Here a_1 and a_2 are the frozen and equilibrium sound speeds, respectively, and are given by

$$a_1^2 = (C_{p1}/C_{v1})(p/\rho), \quad a_2^2 = (C_p/C_v)(p/\rho). \tag{11}$$

Now, we would anticipate, on physical grounds, that the presence of relaxation effects in a gas flow will serve to make 'detailed' changes in the flow pattern when compared with a corresponding inert-gas problem, but will not change the 'orders of magnitude' of quantities involved. For example, in the case of the linearized reacting gas flow round a sharp corner, the pressure coefficient on the surface is found to vary from $-2\theta/B_f$ to $-2\theta/B_e$, where θ is the turning angle and B_f and B_e are the usual Ackeret Mach-number factors based on the frozen and equilibrium sound speeds, respectively. The variation from a B_f to a B_e type of factor represents a 'detailed' change of pressure coefficient, but its 'order of magnitude' remains at the value θ. There is no reason to suspect that this situation will alter when we come to consider the three-dimensional situation involved in the present slender pointed body problem, and accordingly we accept, as a general guide, the orders of magnitude given in Ward's (1949) paper. It should be remarked at this stage that we are only going to deal with steady-flow problems in what follows. Henceforth, therefore, the operator D/Dt becomes synonymous with $\mathbf{q}\,.\,\mathrm{grad}$. Thus we shall follow the procedure, which is rigorously justifiable in the case of inert gas flow past slender bodies (see, for example, Lighthill 1945), of linearizing the basic differential equation but keeping appropriate non-linear terms in the relation between pressure and velocity.

It has been shown by Vincenti (1959) and Moore & Gibson (1960) that the linear approximation in a relaxing or reacting gas flow is consistent with the existence of a velocity potential. Defining a disturbance potential ϕ such that

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial r}, \quad w = \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \tag{12}$$

where u, v, w are the disturbance velocity components in a cylindrical polar co-ordinate system (x, r, θ), the linearized version of (10) becomes

$$\lambda\frac{\partial}{\partial x}\left\{(1-M_f^2)\frac{\partial^2\phi}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right\}+(1-M_e^2)\frac{\partial^2\phi}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0, \tag{13}$$

in which

$$\lambda = \tau U, \quad \tau = C_{p1}\tau'/C_p, \tag{14}$$

where U is the free-stream velocity, directed along the x-axis, λ is the relaxation length, and M_f and M_e are the frozen and equilibrium Mach numbers, namely,

$$M_f = U/a_{1\infty}, \quad M_e = U/a_{2\infty}. \tag{15}$$

(Suffix ∞ denotes a free-stream, or undisturbed, value.)

In accordance with the remarks made above, the pressure coefficient is written as

$$\mathscr{C}_P = \frac{2(p-p_\infty)}{\rho_\infty U^2} \simeq -\frac{2}{U}\frac{\partial \phi}{\partial x} - \frac{1}{U^2}\left[\left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2\right] \tag{16}$$

(see, for example, Ward 1949). The pointed nose of the body is assumed to lie at $x = 0$.

3. The general solution

The general solution of equation (13) can most easily be obtained by taking the exponential Fourier transform. Thus we define

$$\Phi = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \phi\, e^{i\omega x}\, dx. \tag{17}$$

Assuming that ϕ, $\partial\phi/\partial x$ and $\partial^2\phi/\partial x^2$ all vanish asymptotically with increasing distance upstream and downstream from the body, it is found that Φ satisfies the following differential equation:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \omega^2(1-M_f^2)\left\{\frac{1-M_e^2}{1-M_f^2} - i\omega\lambda\right\}\{1-i\omega\lambda\}^{-1}\,\Phi = 0. \tag{18}$$

Expressing Φ as a Fourier series in θ, we find that each coefficient of $\cos n\theta$ or $\sin n\theta$ is the solution of a modified Bessel equation of order n. In fact we can write the solution *formally* as

$$\Phi(r,\theta;\,\omega) = A_0(\omega)\,K_0\{r\omega(1-M_f^2)^{\frac{1}{2}}Z\} + \sum_{n=1}^{\infty} A_n(\omega)\,K_n\{r\omega(1-M_f^2)^{\frac{1}{2}}Z\}\cos(n\theta+\epsilon_n), \tag{19}$$

where A_n $(n=0,1,2,\ldots)$ and $\epsilon_n (n=1,2,\ldots)$ are constants which are to be evaluated from the boundary conditions on the body. Also,

$$Z = \left\{\omega + i\lambda^{-1}\left(\frac{1-M_e^2}{1-M_f^2}\right)\right\}^{\frac{1}{2}}\{\omega+i\lambda^{-1}\}^{-\frac{1}{2}}. \tag{20}$$

Modified Bessel functions of the first kind (i.e. I_n) are rejected in the solution for Φ because they are found not to represent the proper type of 'outgoing' wave system.

In interpreting equation (19), we must distinguish between the following three cases:

(i) $M_f < M_e < 1$: *the subsonic case.* Here we shall write

$$(1-M_f^2)^{\frac{1}{2}} = \beta_f, \quad (1-M_e^2)/(1-M_f^2) = \beta^2 \; (<1). \tag{21}$$

In this case we must write the argument of the Bessel functions as

$$r\beta_f\,|\omega|\,Z, \tag{22}$$

where

$$Z = (\omega + i\lambda^{-1}\beta^2)^{\frac{1}{2}}\,(\omega+i\lambda^{-1})^{-\frac{1}{2}}, \tag{23}$$

and invert the transform along the real ω-axis.

(ii) $M_f < 1 < M_e$: *the transonic case.* In place of (21), we must now write

$$(1-M_f^2)^{\frac{1}{2}} = \beta_f, \quad (M_e^2-1)/(1-M_f^2) = \bar{\beta}^2. \tag{24}$$

Note that $0 < \bar{\beta}^2 < \infty$. The proper behaviour of the solution is then assured if inversion takes place on a contour indented to pass above the singularity at $\omega = 0$, and beginning and ending at $\omega = \infty e^{i\pi}$ and $+\infty$ respectively. The argument of the Bessel functions is now written as

$$r\beta_f \omega Z', \tag{25}$$

where
$$Z' = \{\omega - i\lambda^{-1}\bar{\beta}^2\}^{\frac{1}{2}}\{\omega + i\lambda^{-1}\}^{-\frac{1}{2}}. \tag{26}$$

(iii) $M_e > M_f > 1$: *the supersonic case*. When these Mach number conditions are satisfied, we write

$$(M_f^2 - 1)^{\frac{1}{2}} = B_f, \quad (M_e^2 - 1)/(M_f^2 - 1) = B^2\ (>1). \tag{27}$$

The argument of the Bessel functions is now written as

$$e^{i\frac{1}{2}\pi} r\omega B_f Z'', \tag{28}$$

where
$$Z'' = \{\omega + iB^2\lambda^{-1}\}^{\frac{1}{2}}\{\omega + i\lambda^{-1}\}^{-\frac{1}{2}}. \tag{29}$$

Proper behaviour of the solution is guaranteed if we use the inversion contour described in case (ii).

In dealing with slender bodies we can now approximate equation (19) near the body by retaining only the first terms in the series expansions of the Bessel functions. Then it is found that

$$\Phi(r,\theta;\omega) \simeq -A_0\{C + \log[\tfrac{1}{2}(r|1 - M_f^2|^{\frac{1}{2}}\omega Z)]\} + \frac{1}{2}\sum_{n=1}^{\infty}(n-1)!\,A_n\,2^n(r|1 - M_f^2|^{\frac{1}{2}}\omega Z)^{-n}\cos(n\theta + \epsilon_n), \tag{30}$$

where $C = 0{\cdot}5772\ldots$ is Euler's constant. The terms written formally as ωZ in equation (30) must be interpreted in the forms given above in each of the cases (i), (ii) and (iii). Following Ward (1949), we can now define the complex variable ζ in cross-flow planes, i.e. $\zeta = re^{i\theta}$, whence equation (30) is equivalent to

$$\Omega = \phi + i\psi = a_0(x)\log\zeta + b_0(x) + \sum_{n=1}^{\infty} a_n(x)\,\zeta^{-n}. \tag{31}$$

The quantities a_0, b_0 and a_n $(n = 1, 2, 3, \ldots)$ are defined as follows (using the symbol $\supset$ to mean 'has the Fourier transform'):

$$\left.\begin{aligned} a_0(x) \supset -A_0(\omega), \quad b_0(x) \supset -A_0(\omega)\{C + \log[\tfrac{1}{2}(|1 - M_f^2|^{\frac{1}{2}}\omega Z)]\}, \\ a_n(x) \supset \tfrac{1}{2}(n-1)!\,A_n(\omega)\,2^n(|1 - M_f^2|^{\frac{1}{2}}\omega Z)^{-n}\exp(i\epsilon_n). \end{aligned}\right\} \tag{32}$$

The quantity $\Omega - b_0$ is the same harmonic function in both this case and in Ward's, implying in the present case that the 'incompressible cross-flow' approximation is still valid. In other words, the $a_n(x)$ quantities $(n \geqslant 1)$ can still be found by solving a two-dimensional, incompressible, potential-flow problem with suitable boundary conditions imposed on the surface of a cylinder whose cross-section is that of the body at any chosen station x. To the present order of approximation, these $a_n(x)$ terms are therefore unaffected by relaxation effects.

Since Ward's momentum integral results for over-all forces on the body apply equally well here, it is concluded that both the over-all and local cross-flow forces are unaffected by relaxation in the linearized approximation.

If we omit the base drag term where blunt-based bodies are concerned, the 'inert gas' expression for drag D (which is equally applicable here) is

$$\frac{D}{\frac{1}{2}\rho_\infty U^2} = 4\pi\int_0^L \frac{a_0'}{U}\frac{b_0}{U}dx - 2\pi\left(\frac{a_0 b_0}{U^2}\right)_{x=L} - \frac{1}{U^2}\left(\int_C \phi\frac{\partial\phi}{\partial\nu}d\sigma\right)_{x=L}. \tag{33}$$

In this equation we have written a_0' for the x-derivative of a_0. L is the body length along the x-axis, and the (pointed) nose of the body lies at $x = 0$. In the final integral, $\partial\phi/\partial\nu$ represents the disturbance-potential derivative taken along the outwards normal to the body cross-section contour, and the integral is taken around the perimeter of this cross-section.

As before, we have

$$a_0(x) = US'(x)/2\pi, \tag{34}$$

where $S(x)$ is the body's cross-sectional area. The prime on S denotes differentiation with respect to x. Clearly $S'(0) = 0$ for any streamlined body which begins at $x = 0$, since the body radius is zero there by definition. The additional restriction to pointed nosed bodies, mentioned above, is necessary for the linearizations of the basic equations to be valid. Only in this case will the disturbance velocities remain small compared with U over most of the region adjacent to the body.

The presence of upstream influence effects in the sub- and transonic régimes implies that we should confine our attention to bodies which are also pointed at the tail ($x = L$) in these cases, at least in the absence of *a priori* knowledge about viscous wake patterns. With this additional restriction, equation (33) reduces to

$$\frac{D}{\frac{1}{2}\rho_\infty U^2} = 4\pi\int_0^L \frac{a_0'}{U}\frac{b_0}{U}dx. \tag{35}$$

Equation (33) will only be used in full in the supersonic régime, where the absence of upstream influence permits us to deal with blunt-based bodies.

4. The values of $b_0(x)$

The solution of the relaxation problem is completed once b_0 is found as a function of x. In other words we must invert the Fourier transforms given in equation (32), taking care to treat each of the cases (i), (ii) and (iii) separately. The actual evaluations are carried out in Appendix A and yield the following results:

Case (i)

$$2\pi\frac{b_0}{U} = S'(x)\log\left(\tfrac{1}{2}\beta_f\right) - \frac{1}{2}\int_0^x S''(y)\log(x-y)\,dy + \frac{1}{2}\int_x^L S''(y)\log(y-x)\,dy$$
$$-\frac{1}{2}\int_0^x S'(y)\{\exp[-\beta^2(x-y)\lambda^{-1}] - \exp[-(x-y)\lambda^{-1}]\}\frac{dy}{x-y}. \tag{36}$$

The first two integral terms here are precisely the same as those which occur in the inert gas, subsonic-flow, case (Sears & Adams 1953). The third integral, together with the logarithm term, represents the effects of relaxation. Since an integral of this type is found to arise in other cases, it is worthwhile giving it a special symbol. Thus we write

$$I_\beta = -\frac{1}{2}\int_0^x S'(y)\{\exp[-\beta^2(x-y)\lambda^{-1}] - \exp[-(x-y)\lambda^{-1}]\}\frac{dy}{x-y}. \tag{37}$$

When the integral contains $\bar{\beta}$ or B in place of β we will denote this fact by an appropriate suffix, as in equation (37). In general we shall write it as I_α, where α may be β, $\bar{\beta}$ or B.

A useful alternative form of I_β is found by noting the definition of the exponential integral $\mathrm{Ei}(-a)$: that is,

$$\mathrm{Ei}(-a) = -\int_a^\infty e^{-\sigma}\sigma^{-1}\,d\sigma. \tag{38}$$

Integrating equation (37) by parts with the upper limit equal to $x-\epsilon$, and then taking the limit as $\epsilon \to 0$, we obtain

$$I_\beta = S'(x)\log\beta + \frac{1}{2}\int_0^x S''(y)\left\{\mathrm{Ei}\left(\frac{y-x}{\lambda}\right) - \mathrm{Ei}\left(\beta^2\frac{y-x}{\lambda}\right)\right\}dy. \tag{39}$$

We shall also find some approximate forms of I_β useful in later sections. Thus, when both x/λ and $\beta^2 x/\lambda \to 0$, we find that

$$I_\beta \simeq -\tfrac{1}{2}(1-\beta^2)\,\lambda^{-1}S(x). \tag{40}$$

When both x/λ and $\beta^2 x/\lambda \to \infty$, it can be shown that†

$$I_\beta \simeq S'(x)\log\beta + \tfrac{1}{2}\lambda(\beta^{-2}-1)\,S''(x). \tag{41}$$

Equation (40) is a suitable approximation for near-frozen flow (as $\lambda \to \infty$), and equation (41) is useful in near-equilibrium flow (as $\lambda \to 0$), *provided* β^2 does not become so small that $\beta^2 x/\lambda$ is no longer large. Thus equation (41) cannot be used too near to the beginning of the transonic régime. In the event that $x/\lambda \gg 1$ whilst $\beta^2 x/\lambda \ll 1$, it can be shown that I_β behaves as follows:

$$I_\beta \simeq -\frac{1}{2}\int_0^x S''(y)\log(x-y)\,dy + \tfrac{1}{2}\beta^2\lambda^{-1}S(x) + \tfrac{1}{2}S'(x)\log\lambda - \tfrac{1}{2}CS'(x) - \tfrac{1}{2}\lambda S''(x). \tag{42}$$

This result demonstrates how the solutions begin to break down as one approaches the 'transonic' condition $M_e \to 1$ (identical with $\beta \to 0$) in the limiting case of $\lambda \to 0$. In other words, if $\lambda = 0$ and complete equilibrium prevails, the potential contains a logarithmic singularity in the limit $M_e = 1$. However, when $\lambda \neq 0$, the linear solution continues smoothly up to the condition $M_e = 1$. We shall shortly show that it also passes smoothly and continuously *through* this condition and into the transonic régime.

Case (ii)

$$2\pi\frac{b_0}{U} = S'(x)\log(\tfrac{1}{2}\beta_f) - \int_0^x S''(y)\log(x-y)\,dy$$
$$-\frac{1}{2}\int_x^L S'(y)\exp[-\bar{\beta}^2(y-x)\lambda^{-1}]\frac{dy}{(y-x)} + \frac{1}{2}\int_0^x S'(y)\exp[-(x-y)\lambda^{-1}]\frac{dy}{(x-y)}. \tag{43}$$

† Note that equation (41) always fails in some region sufficiently close to the nose of the body (where $x/L \ll 1$) for any finite value of λ however small. In such regions equation (40) is applicable.

This result should be compared with the subsonic value of b_0 given in equation (36). With use of the notation defined in equations (37) *et seq.*, equation (43) can be rearranged somewhat to read

$$2\pi\frac{b_0}{U} = S'(x)\log(\tfrac{1}{2}\beta_f) - \int_0^x S''(y)\log(x-y)\,dy + \frac{1}{2}\int_0^x S''(y)\log(x-y)\exp[-\bar{\beta}^2(x-y)\lambda^{-1}]\,dy + \frac{1}{2}\int_x^L S''(y)\log(y-x)\exp[-\bar{\beta}^2(y-x)\lambda^{-1}]\,dy - \frac{1}{2}\frac{\bar{\beta}^2}{\lambda}\int_0^L S'(y)\log|x-y|\exp[-\bar{\beta}^2|x-y|\lambda^{-1}]\,dy + I_{\bar{\beta}}. \quad (44)$$

In this form it can fairly readily be seen that equations (36) and (44) become identical in the limits β and $\bar{\beta} = 0$. Thus b_0 passes smoothly into the transonic régime, provided that $0 < \lambda < \infty$. It should be noted that equation (44) only follows from equation (43) if $S'(L) = 0$. We assume $S'(L) = 0$ throughout cases (i) and (ii).

When the conditions x/λ and $\bar{\beta}^2 x/\lambda \ll 1$ are both satisfied (note that this requires M_f not too near to 1), a suitable approximate form of b_0 is

$$2\pi\frac{b_0}{U} \simeq S'(x)\log(\tfrac{1}{2}\beta_f) - \frac{1}{2}\int_0^x S''(y)\log(x-y)\,dy + \frac{1}{2}\int_x^L S''(y)\log(y-x)\,dx - \left(\frac{1+\bar{\beta}^2}{2\lambda}\right)S(x). \quad (45)$$

When the upper limit of the transonic range is approached, so that $\bar{\beta} \to \infty$, we may find a suitable approximate representation of b_0 under the conditions $x/\lambda \ll 1$, $\bar{\beta}^2 x/\lambda \gg 1$, namely

$$2\pi\frac{b_0}{U} \simeq S'(x)\log(\tfrac{1}{2}B_e) - \frac{1}{2}\int_0^x S''(y)\log(x-y)\,dy + \tfrac{1}{2}S'(x)[C - \log\lambda] - \frac{1}{2}\frac{\lambda}{\bar{\beta}^2}S''(x) - \frac{1}{2\lambda}S(x). \quad (46)$$

Provided that we are not now too near to the lower end of the transonic range ($M_e \to 1$), we may be able to satisfy the conditions x/λ and $\bar{\beta}^2 x/\lambda$ both $\gg 1$. In that case, a suitable approximation for b_0 is

$$2\pi\frac{b_0}{U} \simeq S'(x)\log(\tfrac{1}{2}B_e) - \int_0^x S''(y)\log(x-y)\,dy - \tfrac{1}{2}\lambda(1+\bar{\beta}^{-2})S''(x). \quad (47)$$

Finally, we write down the 'near-equilibrium' result at the lower end of the transonic range, i.e. $x/\lambda \gg 1$, $\bar{\beta}^2 x/\lambda \ll 1$. This is

$$2\pi\frac{b_0}{U} \simeq S'(x)\log(\tfrac{1}{2}\beta_f) - \int_0^x S''(y)\log(x-y)\,dy + \frac{1}{2}\int_x^L S''(y)\log(y-x)\,dx - \tfrac{1}{2}S'(x)[C - \log\lambda] - \frac{\bar{\beta}^2}{2\lambda}S(x) - \frac{\lambda}{2}S''(x). \quad (48)$$

Further comment on the behaviour of b_0 in the transonic range will be postponed until we come to consider questions of drag in a later section.

Case (iii)

In the supersonic régime it is found that

$$2\pi\frac{b_0}{U} = S'(x)\log(\tfrac{1}{2}B_f) - \int_0^x S''(y)\log(x-y)\,dy + I_B \tag{49}$$

(see equations (37) *et seq.* for the definition of I_B). Since I_B is identical with I_β, except that B is now written for β, it follows that we can write

$$I_B \simeq \frac{B^2-1}{2\lambda}S(x), \tag{50}$$

when x/λ and B^2x/λ both $\to 0$, and

$$I_B \simeq S'(x)\log B - \tfrac{1}{2}\lambda(1-B^{-2})S''(x), \tag{51}$$

when both x/λ and $B^2x/\lambda \to \infty$.†

The approximation in equation (50) is not suitable if B^2 becomes too large (i.e. if we are too near to $M_f = 1$). In the event that $x/\lambda \ll 1$ whilst $B^2x/\lambda \gg 1$, we find that

$$2\pi\frac{b_0}{U} \simeq S'(x)\log(\tfrac{1}{2}B_e) - \frac{1}{2}\int_0^x S''(y)\log(x-y)\,dy + \tfrac{1}{2}S'(x)[C-\log\lambda] + \frac{\lambda}{2B^2}S''(x) - \frac{1}{2\lambda}S(x). \tag{52}$$

Comparing this result with equation (46), it can be seen that they become identical in the limit $\bar{\beta}\to\infty$, $B\to\infty$. It can be shown that the values of b_0 in the transonic and supersonic cases coincide for all λ when $M_f = 1$, so that a smooth transition from transonic to supersonic states occurs. (See also equations (47) and (49) with (51) when $\bar{\beta}^2\to\infty$ and $B^2\to\infty$.)

5. Drag

Having obtained values for b_0, it is now possible to evaluate the drag of bodies in a relaxing gas flow, making use of Ward's formula (written out in full, except for base drag, in equation (33)). It has been remarked earlier that the sub- and transonic problems must be restricted to bodies pointed at both ends. This condition can be relaxed in the fully supersonic régime. We shall deal with the two classes of body shape separately.

(a) *Doubly-pointed bodies*

Here we can use the simpler formula (35). Writing

$$D = \tfrac{1}{2}\rho_\infty U^2 S_0 \mathscr{C}_D, \tag{53}$$

† The referee has pointed out that equation (51) is not suitable for all classes of body shape; in particular, not for a body whose meridian profile resembles a semi-infinite 'wavy wall'. We therefore require that the bodies must be 'sufficiently smooth', but make no attempt here to investigate the implications of this statement.

where S_0 is some suitable reference area and $\mathscr{C}_D$ is the drag coefficient, we proceed to find values of $\mathscr{C}_D$ in each of the cases (i), (ii) and (iii) treated above.

Case (i)

Using equation (34) for a_0, it can readily be shown that the first two integrals in (36) combine to give a zero contribution to $\mathscr{C}_D$ in all circumstances, as indeed does the first term in (36). Turning to the last integral in (36), namely I_β, we can obtain a compact general result by way of the form of I_β given in (39). The first term in (39) contributes nothing to the final value of $\mathscr{C}_D$ and we are left with

$$S_0\mathscr{C}_D = \frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\left\{\mathrm{Ei}\left(\frac{y-x}{\lambda}\right) - \mathrm{Ei}\left(\beta^2\frac{y-x}{\lambda}\right)\right\} dy\, dx, \tag{54}$$

or, alternatively,

$$S_0\mathscr{C}_D = \frac{1}{4\pi}\int_0^L\int_0^L S''(x)\, S''(y)\left\{\mathrm{Ei}\left(-\frac{|x-y|}{\lambda}\right) - \mathrm{Ei}\left(-\beta^2\frac{|x-y|}{\lambda}\right)\right\} dy\, dx. \tag{54a}$$

It is interesting to compare this result with the wave-drag result for the supersonic flow past doubly-pointed bodies (with no relaxation effects) obtained by Ward and others. (We shall obtain this result in examining case (iii) below—see equation (64).) Here the kernel of the drag integral is made up of the exponential integral terms in place of the simple logarithm. Equation (54) contrasts with the non-relaxing subsonic flow problem, which would yield zero drag since it includes no dissipative mechanism to give rise to a drag. Clearly the $\mathscr{C}_D$ predicted by equations (54) is always positive; for example, in the 'near-frozen' case we may use the approximation (40) to show that

$$S_0\mathscr{C}_D \simeq \frac{(1-\beta^2)}{2\pi\lambda}\int_0^L S'^2(x)\, dx, \tag{55}$$

whilst in 'near-equilibrium' flow (β not too near zero) equation (41) shows that†

$$S_0\mathscr{C}_D \simeq \frac{\lambda(\beta^{-2}-1)}{2\pi}\int_0^L S''^2(x)\, dx. \tag{56}$$

The fact that β is < 1 in subsonic flow then guarantees that $\mathscr{C}_D$ is positive.

Since $\mathscr{C}_D \to 0$ in either limiting case $\lambda \to 0$ or $\lambda \to \infty$, and is positive between these limits, it follows that there must be a maximum value of $\mathscr{C}_D$ somewhere between $\lambda = 0$ and ∞. The value of λ which will make the 'relaxation' drag a maximum will depend on both the body shape and the value of β^2. It does not seem possible to find this value analytically, either in general terms or for a particular body shape, but one would suspect that it should be such as to make L/λ roughly of order one.‡

† In using equation (41) to evaluate drag in the 'near-equilibrium' case, it is implied that the errors introduced into $S_0\mathscr{C}_D$ by ignoring the failure of equation (41) near $x = 0$ are negligible. This is a fairly restrictive condition, as can be appreciated from an examination of the exact value of I_β in the case of a parabolic arc body, treated in this section below. Equation (56) and corresponding expressions derived in cases (ii) and (iii) are sufficient for a discussion of generalities but must be used with care when deriving approximate numerical results.

‡ But see the numerical example later in this section.

In the event that $\beta^2 \to 0$ whilst λ is small, the approximation (55) becomes valid, giving

$$S_0\mathscr{C}_D \simeq -\frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx - \frac{\beta^2}{2\pi\lambda}\int_0^L S'^2(x)\,dx - \frac{\lambda}{2\pi}\int_0^L S''^2(x)\,dx. \tag{57}$$

The appearance of the double integral term in this expression is interesting, since it is precisely one-half of the supersonic wave drag for this class of body shape. In the equilibrium flow limit at $M_e = 1$ ($\beta^2 = 0$), the present theory therefore puts the 'transonic' drag at just this value. Practically of course, the result is incorrect since the original partial differential equation (equation (13)) fails to describe the flow field properly for $M_e = 1$ and $\lambda = 0$. When λ is not zero, we may expect that equation (57) has some validity, however, because in this condition there is a mechanism present for preventing the 'piling-up' of acoustic disturbances which leads to the breakdown of linear solutions when $M_e = 1$ and $\lambda = 0$. A similar situation is encountered in the case $M_f = 1$, $\lambda = \infty$.

Case (ii)

In the transonic case it is best to use the result (43) to find the drag. After some rearrangement, it can be shown that

$$S_0\mathscr{C}_D = -\frac{1}{\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx + \frac{1}{\pi}\int_0^L S''(x)\int_0^x S''(y) \times \mathrm{Ei}\left[\bar{\beta}^2\frac{(y-x)}{\lambda}\right]dy\,dx + \frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\left\{\mathrm{Ei}\left[\frac{y-x}{\lambda}\right] - \mathrm{Ei}\left[\bar{\beta}^2\frac{(y-x)}{\lambda}\right]\right\}dy\,dx. \tag{58}$$

Comparing this with the result for the subsonic case (equation (54)), it can be seen that a term exactly equivalent to $S_0\mathscr{C}_D$ (subsonic) occurs, the only difference being the appearance of $\bar{\beta}$ for β. Now, however, the drag is reinforced by the presence of a full supersonic wave-drag contribution (first integral in equation (58)), combined with a further relaxation term. To distinguish between the various contributions to $\mathscr{C}_D$ in the present case, we shall refer to the first two integral terms in (58) as the 'transonic wave drag', and to the last integral as the 'relaxation drag'. This division is simply for convenience later: note that the 'transonic wave drag' in fact includes relaxation terms. We recall that $0 \leqslant \bar{\beta}^2 \leqslant \infty$.

When $\lambda \to \infty$ and we are not too near to $\bar{\beta} = \infty$, it is found that the transonic wave drag contributes a term

$$\frac{\bar{\beta}^2}{\pi\lambda}\int_0^L S'^2(x)\,dx \tag{59a}$$

to $S_0\mathscr{C}_D$, whilst the relaxation drag part behaves like

$$\frac{1-\bar{\beta}^2}{2\pi\lambda}\int_0^L S'^2(x)\,dx. \tag{59b}$$

The net effect is to give rise to a positive value of $\mathscr{C}_D$, of course, but it is interesting to note that the relaxation-drag contribution falls to zero and then becomes *negative* as $\bar{\beta}$ passes through the value unity. That expression (59*b*) is part of a general result valid for all values of λ can readily be seen from equation (58). The change-over point occurs when

$$U \simeq (a_{1\infty} a_{2\infty})^{\frac{1}{2}}, \tag{60}$$

roughly speaking, if $a_{1\infty}$ and $a_{2\infty}$ do not differ greatly. Meanwhile, the transonic wave drag continues to increase as $\bar{\beta}$ increases, as indeed does the net drag.

Still retaining the assumption $\lambda \to \infty$, but now imagining that $\bar{\beta}^2$ has become so large that $\bar{\beta}^2 L/\lambda \to \infty$, we find that the two parts of $S_0 \mathscr{C}_D$ behave like

$$-\frac{1}{\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx - \frac{\lambda}{\pi\bar{\beta}^2}\int_0^L S''^2(x)\,dx \tag{61a}$$

and

$$+\frac{\lambda}{2\pi\bar{\beta}^2}\int_0^L S''^2(x)\,dx + \frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx + \frac{1}{2\pi\lambda}\int_0^L S'^2(x)\,dx, \tag{61b}$$

respectively. The transonic wave drag in (61 *a*) continues to increase as $\bar{\beta}$ increases to its final value in the present conditions and, in line with the comments made above about the relaxation drag, this part (61 *b*) is found to be negative. The double integral term in (61 *a*) is the full supersonic wave drag for a doubly pointed body in a non-relaxing gas flow (essentially a positive contribution to $S_0 \mathscr{C}_D$). It seems clear from the results (59 *a*) and (61 *a*) that the transonic wave drag increases steadily from zero at the subsonic ($M_e = 1$) end of the transonic range up to the full supersonic value at its upper end ($M_f = 1$), a result which we will confirm below in the other extreme of $\lambda \to 0$.

We observe also (at least when $\lambda \to \infty$) that the net value of $S_0 \mathscr{C}_D$ on passing out of the transonic régime ($\bar{\beta} = \infty$) is a little greater than one half of the full supersonic wave drag without relaxation effects. However, the linear solution breaks down at $M_f = 1$ when $\lambda \to \infty$, on account of the logarithmic singularity arising in b_0 (see equation (46)).

When $\lambda \to 0$, suitable approximate forms of the transonic wave and relaxation drags are, in the former case, exactly equation (61 *a*) and, in the latter case,

$$\frac{\lambda}{2\pi}(\bar{\beta}^{-2} - 1)\int_0^L S''^2(x)\,dx, \tag{62}$$

provided $\bar{\beta}$ is not too near to zero. In the event that $\bar{\beta}^2 L/\lambda \to 0$ despite L/λ being $\gg 1$, we find that the two parts of $\mathscr{C}_D$ are given by equation (59 *a*) for the transonic wave drag, and

$$-\frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx - \frac{\bar{\beta}^2}{2\pi\lambda}\int_0^L S'^2(x)\,dx - \frac{\lambda}{2\pi}\int_0^L S''^2(x)\,dx \tag{63}$$

for the relaxation drag. Note that the sum of expressions (59 *a*) and (63) gives a value of $S_0 \mathscr{C}_D$ exactly equal to that in equation (57) when both β^2 and $\bar{\beta}^2$ are zero. When $\beta^2 = \bar{\beta}^2 \neq 0$, however, the net transonic drag is greater than the subsonic value, as we should expect. That a similar state of affairs arises when λ is large follows from comparison of equation (55) with the sum of expressions (59 *a*) and (59 *b*).

An explanation of the negative contribution to the over-all drag which is provided by the relaxation term for $\bar{\beta}^2 > 1$ will be deferred until the section on pressure distributions which is to follow, but we remark here that it appears to be a characteristic of the pointed-tail body shape required in the sub- and transonic régimes by the present theory.

Case (iii)

For the fully supersonic régime, the relevant value of b_0 is found from equation (49), and it follows that

$$S_0\mathscr{C}_D = -\frac{1}{\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx$$
$$-\frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\left\{\mathrm{Ei}\left[B^2\frac{(y-x)}{\lambda}\right]-\mathrm{Ei}\left[\frac{y-x}{\lambda}\right]\right\}dy\,dx. \quad (64)$$

In this equation, $B^2 > 1$ in the 'relaxation' drag integral. The contribution made by this integral to the net $\mathscr{C}_D$ value is negative, as we shall confirm below. The first integral in equation (64) is the full supersonic wave drag for a non-relaxing gas flow.

When $\lambda \to \infty$ and we are not too near to $M_f = 1$, the relaxation drag is approximately

$$-\frac{(B^2-1)}{2\pi\lambda}\int_0^L S'^2(x)\,dx, \quad (65)$$

whilst, when $\lambda \to 0$, it contributes a term

$$-\lambda\frac{(1-B^{-2})}{2\pi}\int_0^L S''^2(x)\,dx \quad (66)$$

for all values of B^2.

If $\lambda \to \infty$, yet B^2 becomes so large that still $B^2L/\lambda \gg 1$, we find that the relaxation drag behaves like

$$\frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx+\frac{\lambda}{2\pi B^2}\int_0^L S''^2(x)\,dx+\frac{1}{2\pi\lambda}\int_0^L S'^2(x)\,dx. \quad (67)$$

When $B^2 = \infty$, equation (67) plus the full supersonic wave drag gives a result for $S_0\mathscr{C}_D$ which agrees with the sum of expressions (61*a*) and (61*b*) for $\bar{\beta}^2 = \infty$. The same agreement occurs in the near-equilibrium state, for putting $\bar{\beta}^2 = \infty$ in the sum of equations (61*a*) and (62) gives exactly the result found by adding equation (66) with $B^2 = \infty$ to the full supersonic wave drag.

In the supersonic régime, the effect of relaxation is such as to give $\mathscr{C}_D$ values which are always less than the non-relaxing wave drag on doubly-pointed bodies.

To conclude this section on doubly-pointed bodies, we consider a specific example, namely the body whose meridian profile is a parabolic arc. We only examine the sub- and supersonic régimes since, for most practical cases, the transonic régime is very narrow and the solutions that we have obtained in this region are of rather more academic than practical interest. We shall write

$$S_0\mathscr{C}_{Dr} = \frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\left\{\mathrm{Ei}\left[\frac{(y-x)}{\lambda}\right]-\mathrm{Ei}\left[\alpha^2\frac{(y-x)}{\lambda}\right]\right\}dy\,dx, \quad (68)$$

where α^2 may be either β^2 or B^2, as the case may be. $\mathscr{C}_{Dr}$ will be referred to as the relaxation drag coefficient. The method of evaluation of the integral in (68) is outlined in Appendix B. Equation (B 12) gives the equation for the meridian profile and equation (B 13) gives the value of $\mathscr{C}_{Dr}$ for this body shape, assuming that the reference area S_0 is equal to the maximum body cross-sectional area. Note that in deriving equation (B 13) the body length L has been set equal to one (no loss of generality is incurred by this), so that λ is the same as τU measured in units of body length.

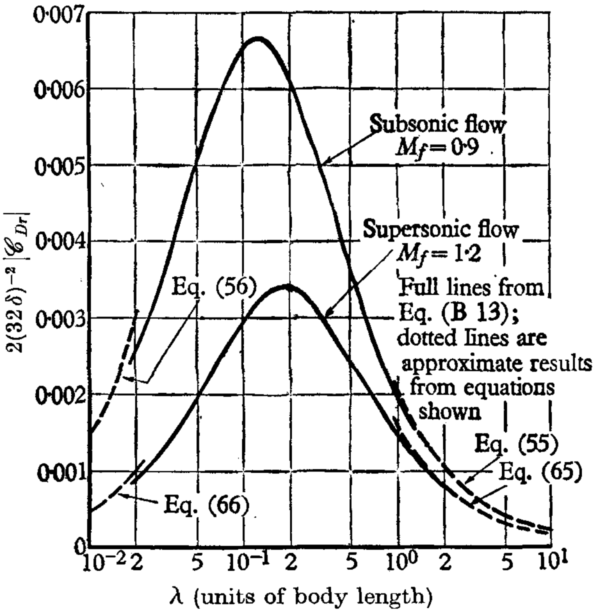

FIGURE 1. Relaxation drag for a parabolic arc body. $(a_{1\infty}/a_{2\infty})^2 = 1·1$.

Figure 1 shows $2(32\delta)^{-2}|\mathscr{C}_{Dr}|$ plotted against λ for two values of α^2, namely 0·574 and 1·327. These values correspond to values for M_f of 0·9 and 1·2 respectively, with the square of the sound-speeds ratio equal to 1·1. This latter value is roughly that which occurs in CO_2 at temperatures somewhat greater than 288 °K. For this linear triatomic molecule, the relaxing internal mode is that involving transverse vibrations and at N.T.P. the relaxation time is about 10 μsec. We note that the maxima for $|\mathscr{C}_{Dr}|$ occur for values of λ between 0·1 and 0·3 body lengths, so that with values of U of about 3×10^4 cm/sec, we find that the body lengths to give this maximum are roughly of order $(0·3/p_a)$ cm, where p_a is the free-stream pressure measured in atmospheres. In this particular case then, even pressures as high as 10^{-1} atm. can lead to maxima in $|\mathscr{C}_{Dr}|$ for body sizes comparable to those found in a number of experimental facilities. It is not possible to state in general terms just how significant $\mathscr{C}_{Dr}$ is in relation to skin friction and form drag, since these are viscous effects and hence outside the scope of the present theory. However, under conditions like those just described for the occurrence of $|\mathscr{C}_{Dr}|_{\max}$, the Reynolds number based on body length should be of order 10^6. Thus the laminar-friction drag coefficient based on body surface area will be of order

10^{-3}, which, for a slender body, is roughly a $\mathscr{C}_D$ of $10^{-3}\delta^{-1}$ when the reference area is $\pi\delta^2$ (as in the definition of $\mathscr{C}_{Dr}$). Allowing for the crudeness of the estimated friction drag coefficient, it is seen that $|\mathscr{C}_{Dr}|_{\max}$ *may* be comparable with it in some circumstances. For a well streamlined body, on which friction drag predominates, it seems safe to conjecture that relaxation drag may cause significant increases in over-all drag in subsonic flow. Clearly each special case must be examined in detail, but the foregoing estimates do tend to suggest that such examination is worthwhile.

In supersonic flow, of course, the drag picture is dominated by the appearance of wave drag. For the case of the parabolic-arc body, the wave-drag coefficient $\mathscr{C}_{Dw}$ is found to be given by
$$2(32\delta)^{-2}\mathscr{C}_{Dw} = \tfrac{1}{12},$$
and clearly this is much larger than the values of $|\mathscr{C}_{Dr}|$ shown in figure 1. We note that $|\mathscr{C}_{Dr}|$ may still be comparable with friction drag coefficients.

In conclusion we note that the amplitude of the effects of relaxation will increase as α^2 increases. Broadly speaking, this implies that the largest effects will arise in the transonic régime, but, for obvious reasons, the present linearized theory cannot be used to obtain numerical results of convincing accuracy in that Mach number range. α^2 is increased at any M_f value by increasing the speeds of sound ratio a_1/a_2. A maximum value of $(a_1/a_2)^2$ (namely 25/21) occurs for rotational relaxation in a diatomic gas, but we note that much larger values can arise when chemical reactions are involved (e.g. see Clarke 1958).

(*b*) *Blunt based bodies*

Confining ourselves to the supersonic régime in order to be able to deal with this particular class of body shapes, equations (33), (34) and (49) combine to show that

$$\begin{aligned} S_0\mathscr{C}_D = &-\frac{1}{\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx + \frac{S'(L)}{2\pi}\int_0^L S''(y)\log(L-y)\,dy \\ &+\frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\left\{\mathrm{Ei}\left(\frac{y-x}{\lambda}\right)-\mathrm{Ei}\left(B^2\frac{y-x}{\lambda}\right)\right\}dy\,dx \\ &-\frac{S'(L)}{4\pi}\int_0^L S''(y)\left\{\mathrm{Ei}\left(\frac{y-L}{\lambda}\right)-\mathrm{Ei}\left(B^2\frac{y-L}{\lambda}\right)\right\}dy - \frac{1}{U^2}\left(\int_C \phi\frac{\partial\phi}{\partial\nu}\,d\sigma\right)_{x=L}, \end{aligned} \tag{69}$$

where $\mathscr{C}_D$ is defined as in (53) above. The final integral here can only be evaluated once the particular body shape and attitude is given. For example, it is this integral which determines the effects of body incidence on $\mathscr{C}_D$. Since the cross-flow problem is unaffected by relaxation to the present order of approximation, we will have no need to consider questions of body attitude, for the results will be identical with those already obtained by Ward. Thus we can confine our attention to the 'zero-incidence' behaviour of the integral in question, incidence being measured from the body position giving zero cross-wind force.

In particular, we examine the body of revolution, for which the normal derivative $\partial\phi/\partial\nu$ becomes simply $\partial\phi/\partial r$, and the boundary condition gives
$$\partial\phi/\partial r = UR'(x), \tag{70}$$

on the contour C. We write $R(x)$ for body radius, the prime denoting differentiation with respect to x, as usual. In these circumstances, the disturbance potential is

$$\phi(x,r) = U\frac{S'(x)}{2\pi}\log r + b_0(x), \tag{71}$$

as there is no dependence on the angle θ, and it then follows that

$$-\frac{1}{U^2}\left(\int_C \phi\frac{\partial\phi}{\partial\nu}d\sigma\right)_{x=L} = -\frac{S'^2(L)}{2\pi}\log R(L) - S'(L)\frac{b_0(L)}{U}. \tag{72}$$

When we take the value of b_0 from equation (49) and use the form of I_B given in equation (39), the sum of wave plus relaxation drag for a blunt-based body of revolution at zero incidence is found to be

$$\begin{aligned} S_0\mathscr{C}_D = &-\frac{S'^2(L)}{2\pi}\log\{\tfrac{1}{2}B_e R(L)\} - \frac{1}{\pi}\int_0^L S''(x)\int_0^x S''(y)\log(x-y)\,dy\,dx \\ &+\frac{S'(L)}{\pi}\int_0^L S''(y)\log(L-y)\,dy \\ &+\frac{1}{2\pi}\int_0^L S''(x)\int_0^x S''(y)\left\{\mathrm{Ei}\left(\frac{y-x}{\lambda}\right) - \mathrm{Ei}\left(B^2\frac{y-x}{\lambda}\right)\right\}dy\,dx \\ &-\frac{S'(L)}{2\pi}\int_0^L S''(y)\left\{\mathrm{Ei}\left(\frac{y-L}{\lambda}\right) - \mathrm{Ei}\left(B^2\frac{y-L}{\lambda}\right)\right\}dy. \end{aligned} \tag{73}$$

Notice the appearance of the factor B_e in the first term of (73). Because this term arises naturally in the expression for $\mathscr{C}_D$ we shall take the equilibrium state as the basis for comparison in what follows. The first three terms in (73) will be referred to as the drag occurring in a non-relaxing gas; the last two integrals therefore summarize the relaxation effects.

In the near equilibrium state, these relaxation terms behave like

$$-\lambda\frac{(1-B^{-2})}{2\pi}\left\{\int_0^L S''^2(x)\,dx - S'(L)\,S''(L)\right\}, \tag{74}$$

to a first order in λ. When $\lambda \to \infty$ (and B^2 is not too large), they are approximately

$$\frac{S'^2(L)}{2\pi}\log B - \frac{B^2-1}{2\pi\lambda}\int_0^L S'^2(x)\,dx. \tag{75}$$

This last result confirms that, in the limit $\lambda \to \infty$, the value of $S_0\mathscr{C}_D$ becomes the one appropriate to a gas flow at the frozen Mach number M_f, since the logarithm here combines with that in the non-relaxing gas flow to give a term

$$-S'^2(L)\,(2\pi)^{-1}\log\{\tfrac{1}{2}B_f R(L)\}.$$

This result also indicates that $S_0\mathscr{C}_D$ is always somewhat less than the 'frozen Mach number' value.

It is not so easy to deduce what is happening to $S_0\mathscr{C}_D$ in the near-equilibrium state, since its behaviour will depend on the relative magnitude of the terms $S'(L)\,S''(L)$ and the integral in (74). We may expect that the behaviour depends on body shape, and it seems reasonable to use the parabolic arc body of the previous numerical example to indicate roughly what this may be. If this body shape is assumed to terminate at $x = \frac{3}{4}L$ we find that the term in the brackets $\{\}$ in (74) has the value $+(32\pi\delta^2)^2 L(81/640)$; when the body ends at $x = \frac{1}{2}L$, it has the value $+(32\pi\delta^2)^2 L(1/10)$ and when the body ends at $\frac{1}{4}L$, it has the value $-(32\pi\delta^2)^2 L(3/32)$. In the last case, where $S'(x)$ is definitely positive at the base, the drag is greater than the non-relaxing gas value; but, somewhere between this condition and the one which gives $S'(x) = 0$ at the base, the relaxation drag increment passes through zero and becomes negative.

To summarize then, we always expect to find drag less than the frozen Mach number value. Depending on the body shape and the values of λ and B^2, we may even find drag somewhat less than the equilibrium-flow value. As λ increases in such a case, it seems reasonable to suppose that the net drag will eventually begin to increase, become equal to the equilibrium value once more (this time at a finite, non-zero value of λ), and thence lie between the equilibrium and frozen-flow values of $\mathscr{C}_D$ for all higher values of λ. It is worth noting the difference between the (quite complicated) drag behaviour deduced here and that found by Vincenti (1960) for the wavy wall. In the latter case, $\mathscr{C}_D$ always lies between the equilibrium and frozen Mach number values in the supersonic régime, probably on account of the multiplicity of reflected disturbances emanating from upstream regions of the flow.

The simplest example of a blunt-based body is the right-circular cone and we proceed to give some numerical results for this shape. With a body length of unity, we take

$$R(x) = \delta x \tag{76}$$

for the cone. Whence it follows that $S''(x) = 2\pi\delta^2$. Some of the results in Appendix B can be used to evaluate the integrals occurring in equation (73), and it is found that the relaxation drag coefficient $\mathscr{C}_{Dr}$ is given by

$$\delta^{-2}\mathscr{C}_{Dr} = \lambda^2(1-B^{-4}) - \lambda^2(e^{-1/\lambda} - B^{-4}e^{-B^2/\lambda}) - \lambda(e^{-1/\lambda} - B^{-2}e^{-B^2/\lambda}) + E_1(1/\lambda) - E_1(B^2/\lambda). \tag{77}$$

(The notation is explained in Appendix B.) The wave-drag contribution to the total $\mathscr{C}_D$ is given by

$$\delta^{-2}\mathscr{C}_{Dw} = 2\log(2/B_e\delta) - 1, \tag{78}$$

where we write $\mathscr{C}_{Dw}$ for the wave-drag coefficient. All drag coefficients have been based on a value $S_0 = \pi\delta^2$.

Figure 2 shows $\delta^{-2}\mathscr{C}_{Dr}$ plotted against λ in units of body length for the value $B^2 = 1{\cdot}223$. With the square of the speeds of sound ratio equal to $1{\cdot}1$, this corresponds to an M_e of $\sqrt{2}$.

The presence of the factor $\log(2/B_e\delta)$ in $\mathscr{C}_{Dw}$ makes the ratio of $\mathscr{C}_{Dr}$ to wave drag dependent on the thickness ratio of the body. For example, when $\delta = \tan 5°$, $\delta^{-2}\mathscr{C}_{Dw} = 5{\cdot}24$. We see that, as in the case of the parabolic arc body at supersonic speeds, the contribution made by relaxation effects is very small in comparison with the wave drag.

The cone is a further example of the type of body for which $\mathscr{C}_{Dr}$ is always of one sign for any value of λ (compare the 'one-quarter' parabolic-arc body mentioned above).

To conclude this section on drag, we may summarize the results briefly by saying that the effects of relaxation depend quite strongly on actual body shape. It seems likely that such effects are comparable with skin-friction drag but do not seem to approach the magnitude of wave-drag coefficients.

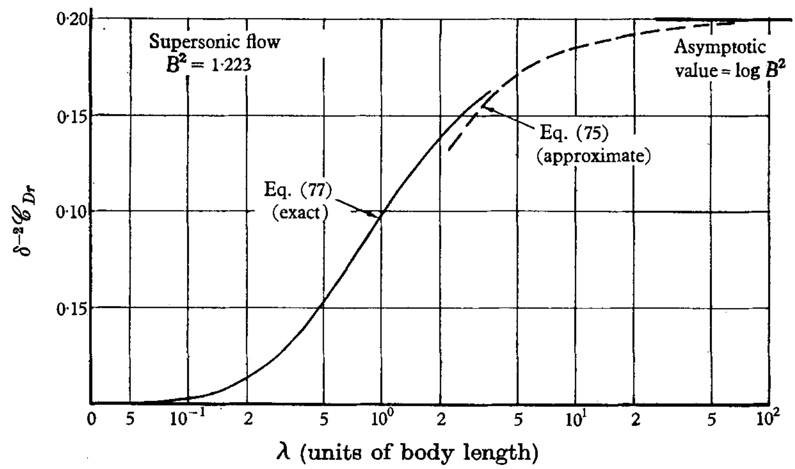

FIGURE 2. Relaxation drag for a cone.

6. The pressure coefficient

The pressure coefficient is defined in equation (16), which also shows the approximate relation between $\mathscr{C}_P$ and the disturbance potential in the present, linear, theory. We are rather more interested in the differences which arise as a result of relaxation effects, and it is clear from equations (16) and (31) that this only involves the difference between the relevant $b_0(x)$ functions. In fact, writing $(\mathscr{C}_P)_r$ and $(\mathscr{C}_P)_e$ for the pressure coefficients with and without relaxation effects, respectively, with a similar suffix notation for the b_0 functions, it is found that

$$\Delta(\mathscr{C}_P) = (\mathscr{C}_P)_r - (\mathscr{C}_P)_e = -2U^{-1}(b'_{0r} - b'_{0e}). \tag{79}$$

(The prime on b_0 indicates differentiation with respect to x.) To the present approximation, equation (79) is true for all shapes and attitudes which come within the classification 'slender body'. $(\mathscr{C}_P)_e$ is chosen to be the pressure coefficient under full equilibrium conditions (i.e. the singular case, $\lambda = 0$).

We shall not consider the transonic régime here and, this being so, it follows that

$$\Delta\mathscr{C}_P = -\frac{1}{\pi}\{I'_\alpha - S''(x)\log\alpha\}, \tag{80}$$

where α can be either β or B, as the case may be. (Equation (80) follows from the results for b_0 given in (36) and (49), together with the definition of I_α in (37) and (39).) Some of the approximate forms of I_α derived in §4 can now be used to indicate roughly how $\Delta\mathscr{C}_P$ behaves in different situations. (One can verify that the derivative of the approximate form of I_α is the same as the approximate form of the derivative.) Thus, when $x/\lambda \to 0$, equations (40) and (50) show that

$$\pi\Delta\mathscr{C}_P \simeq \tfrac{1}{2}(1-\alpha^2)\,\lambda^{-1}S'(x) + S''(x)\log\alpha, \tag{81}$$

and when $x/\lambda \to \infty$, equations (41) and (51) show that

$$\pi\Delta\mathscr{C}_P \simeq \tfrac{1}{2}\lambda(1-\alpha^{-2})\,S'''(x). \tag{82}$$

Since equation (81) will be a valid approximation for sufficiently small x for any value of λ, other than the singular value $\lambda = 0$, we infer that $\mathscr{C}_P$ at the nose of the body always equals the frozen flow value. The only exception occurs in the singular case $\lambda = 0$, when of course $\Delta\mathscr{C}_P = 0$ everywhere, as can be seen from equation (82). In general then, a relaxation zone, starting from the frozen $\mathscr{C}_P$ value begins to form at the nose of the body and this is true whether the flow be subsonic or supersonic. Equation (82) indicates that the cone, for which $S'''(x) = 0$, is rather a special case in so far as no relaxation effects will occur in the near equilibrium state ($\lambda \to 0$) in regions where $x/\lambda \gg 1$.

It is interesting to note that the present theory indicates no 'upstream influence' of relaxation effects on $\mathscr{C}_P$ in subsonic flow. Indeed the integral term I_β (and hence its derivative with respect to x) vanishes for all $x < 0$. Since $S''(x)$ is also zero for $x < 0$, it follows from equation (80) above that $\mathscr{C}_P$ has its equilibrium value upstream of $x = 0$. That upstream influence on $\mathscr{C}_P$ does exist in subsonic flow follows from the behaviour of the second integral in (36). When $x < 0$, this integral can be written as

$$\frac{1}{2}\int_0^L S''(y)\log(y+|x|)\,dy \neq 0.$$

It will shortly be shown that similar conclusions do not follow about the relaxation effects on the translational and internal mode temperatures in subsonic flow. It would seem reasonable to conclude that the lack of upstream relaxation effect upon $\mathscr{C}_P$ in subsonic flow is a characteristic of the linear theory, and that it would therefore be safer to say that such effect as might exist for $\mathscr{C}_P$ is at most of second order. In supersonic flow there is no such difficulty since upstream influence (quite properly) does not exist at all.

Returning to the approximate $\Delta\mathscr{C}_P$ values in equations (81) and (82), we observe first the change in sign of the terms involving α in the two Mach number régimes. These changes of sign are consistent with the relaxation drag behaviour in subsonic and supersonic flow and indicate in fact how this comes about. For subsonic flow with $L/\lambda \to 0$ (remembering that $\alpha = \beta < 1$), equation (81) shows that it is the term in $S'(x)$ which produces the relaxation drag, since that in $S''(x)$ has zero net effect on the drag of the doubly pointed bodies required in the subsonic flow case. In supersonic flow under the condition $L/\lambda \to 0$ the influence of this $S''(x)$ term begins to be felt on the blunt based bodies permitted in this régime.

The influence of body shape on the pressure shifts arising from relaxation effects is clear from equations (81) and (82). It is also clear that each particular case must be treated on its merits. What is *not* quite so clear is *how* these effects are brought about. The most plausible explanation seems to be just the one advanced by the author (Clarke 1960*a*) to explain the much simpler behaviour in the relaxation zone behind a sharp corner in supersonic flow, namely that the outgoing pressure waves generated by the body are reflected back towards the surface by the relaxation-generated vorticity distribution. It is not surprising to find that the signs and magnitudes of these reflected waves depend on body shape and Mach number régime. To conclude this section, we remark that past experience has indicated that a linearized theory is capable of yielding quite accurate estimates of the superimposed effects of relaxation, even though the basic quantities (like $(\mathscr{C}_P)_e$ for example) are not so well predicted. Also, it must be emphasized that the solutions discussed here, and indeed in the section to follow, are only valid on and near the body surfaces.

7. The temperature variations

In order to find the variations of the internal mode and translational temperatures, we first note that the energy equation can be written in terms of the specific enthalpy h and pressure as follows:

$$\rho\frac{Dh}{Dt}-\frac{Dp}{Dt}=0.$$

The enthalpy h can be written as $C_{p1}T_1+C_{v2}T_2$ for present purposes, and, consistent with the previous linearizations, ρ can be replaced by its undisturbed stream value ρ_∞. Then this equation can be integrated to give the approximate result

$$C_{p1}\Delta T_1+C_{v2}\Delta T_2 \simeq \rho_\infty^{-1}(p-p_\infty). \tag{83}$$

We have written

$$\Delta T_1 = T_1-T_{1\infty}, \quad \Delta T_2 = T_2-T_{1\infty}, \tag{84}$$

since $T_{2\infty}=T_{1\infty}$. It follows from equations (8) and (83) that

$$\tau\frac{D\Delta T_2}{Dt}+\Delta T_2 \simeq \frac{U^2\mathscr{C}_P}{2C_p}. \tag{85}$$

Putting $D/Dt \simeq U\partial/\partial x$, we readily find that

$$\Delta T_2 = \frac{U}{2\tau C_p}\int_{-\infty}^{x}\mathscr{C}_P\, e^{-(x-y)/\lambda}\,dy \tag{86}$$

in subsonic flow. In supersonic flow the lower limit can be replaced by zero. Using equation (8), and after some rearrangement, we find that

$$\Delta T_1 = \frac{U^2}{2C_p}\mathscr{C}_P+\left(\frac{C_{v2}}{C_{p1}}\right)\frac{U^2}{2C_p}\int_{-\infty}^{x}\frac{\partial\mathscr{C}_P}{\partial y}\,e^{-(x-y)/\lambda}\,dy. \tag{87}$$

Integration of equation (86) by parts and combination of the result with equation (87) shows that

$$T_1-T_2 = \frac{U^2}{2C_{p1}}\int_{-\infty}^{x}\frac{\partial\mathscr{C}_P}{\partial y}\,e^{-(x-y)/\lambda}\,dy. \tag{88}$$

This last result shows at once that an upstream influence of the relaxation effects exists for the temperatures in subsonic flow, since $\partial\mathscr{C}_P/\partial y \neq 0$ for $y<0$ in that case.

It is important to observe that the temperatures on and near the body are functions of the whole pressure coefficient. In particular this fact is significant when we examine the difference between T_1 and T_2. In the singular case $\lambda = 0$, the difference vanishes, of course, but we may use the other limiting case of $\lambda \to \infty$ to illustrate the practical situation. For the near-frozen state we may write $T_2 \simeq T_{1\infty}$, whence $T_1 - T_2 \simeq \Delta T_1$ and we find that this difference in the limit is just $\frac{1}{2}U^2\mathscr{C}_P/C_{p1}$, (according to either equation (87) or equation (88)). Thus the difference between T_1 and T_2 depends on the body shape and attitude in the same way as $\mathscr{C}_P$. Clearly this behaviour occurs for all non-zero λ, but we expect to find the amplitude of the difference decreasing in a general way with decreasing λ. It may be of some significance however that relatively large differences between T_1 and T_2 could arise locally due to some peculiarity in body shape. The results presented above enable calculations of T_1 and T_2 to be made (within the confines of the slender-body approximation), and these values could be used to estimate temperatures at the edge of the thermal boundary layer. The latter statement is made with reservations, since it may happen that the concept of the boundary layer, and the 'body-plus-displacement-thickness' treatment which is implied by it, begin to lose validity at pressures (and hence, broadly speaking, Reynolds numbers) low enough to make λ/L sufficiently large. (The question of relaxation effects on such a flow is an interesting one, but outside our present scope. From the foregoing results we might expect that viscous and relaxation effects will be comparable over large regions of the flow field.) Clearly any particular case must be examined on its merits, but we may quote the case of carbon dioxide, for which appreciable relaxation effects are present at pressures high enough for the low-density aspects of viscous flow (such as slip phenomena) to be insignificant. In other words, the mean-free molecular path will be small compared with the boundary-layer thickness in this case. The non-dimensional group which roughly determines the state of the internal energy mode in a boundary layer is $(\delta_1^2/\tau'\mathscr{D})^{\frac{1}{2}}$, where δ_1 is boundary-layer thickness and $\mathscr{D}$ is the self-diffusion coefficient for the particular (pure) gas in question. Assuming a laminar boundary layer, which seems reasonable in the circumstances, we have $\delta_1 \sim x^{\frac{1}{2}}(\nu/U)^{\frac{1}{2}}$, where ν is the kinematic viscosity. Thus,

$$\left(\frac{\delta_1^2}{\tau'\mathscr{D}}\right)^{\frac{1}{2}} \sim \left(\frac{x}{L}\right)^{\frac{1}{2}}\left(\frac{L}{\tau'U}\right)^{\frac{1}{2}}\left(\frac{\nu}{\mathscr{D}}\right)^{\frac{1}{2}}.$$

The ratio $\nu/\mathscr{D}$ is approximately unity for a number of gases, and it follows that if the external (inviscid) flow is near-frozen it will also be near-frozen in the boundary layer. The appearance of the ratio x/L above guarantees that it *will* be frozen at the nose of the body, but it is significant to notice that it is $(L/\tau'U)^{\frac{1}{2}}$ (or roughly $\lambda^{-\frac{1}{2}}$) which determines boundary-layer behaviour whilst the external flow is governed by the value λ^{-1}.

In cases where near-frozen flow occurs in the layer, the effectiveness of the actual material of the body's surface in accommodating the internal mode becomes important in the determination of energy flux rates (see, for example, Clarke 1960*b*). Thus the interpretation of heat-transfer measurements on bodies in polyatomic gas flows must be approached with caution. Similar arguments apply with even greater force to the case of chemically reacting gas flows, on account of the greater energies involved and the possibility of finding wide ranges of catalytic efficiencies amongst practical materials. Remarks such as these might apply to heat-transfer measurements on, let us say, a slender cone in a typical shock tube. Results like those presented above may be used to estimate the conditions at the edge of the boundary layer in such a case.

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Appendix A

Inversion integrals for the function $b_0(x)$

Case (i)

In the subsonic case, $b_0(x)$ is written as

$$b_0(x) \supset -A_0(\omega)\{C+\log(\tfrac{1}{2}\beta_f|\omega|\,Z\}, \tag{A 1}$$

(see equations (22), (23) and (32)). To accomplish the inversion integration, we write

$$\begin{aligned} b_0(x) \supset -A_0(\omega)\log(\tfrac{1}{2}\beta_f) - i\omega(-A_0)\left\{\frac{C+\log|\omega|}{-i\omega}\right\} \\ +\tfrac{1}{2}(-i\omega+\beta^2/\lambda)(-A_0)\left\{\frac{C+\log(\omega+i\beta^2/\lambda)}{-i(\omega+i\beta^2/\lambda)}\right\} \\ -\tfrac{1}{2}(-i\omega+1/\lambda)(-A_0)\left\{\frac{C+\log(\omega+i/\lambda)}{-i(\omega+i/\lambda)}\right\}. \end{aligned} \tag{A 2}$$

The first term in (A 2) is simply $a_0(x)\log(\frac{1}{2}\beta_f)$. To invert the remainder we make use of the fact that $-i\omega$ is equivalent to the operation $\partial/\partial x$ and treat the remaining parts by means of the Faltung theorem for complex Fourier transforms, which states that

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy \supset F(\omega)\,G(\omega). \tag{A 3}$$

F and G are the transforms of $f(x)$ and $g(x)$. Thus we can deal with the log terms in curly brackets separately, taking care to use the contour described in § 3, case (i).

The first curly bracket term in (A 2) inverts to a function $g_1(x)$, where

$$\begin{aligned} \sqrt{(2\pi)}\,g_1(x) &= \operatorname*{Lim}_{\epsilon\to 0}\left\{\int_\epsilon^\infty (C+\log u)\,e^{ixu}\frac{du}{iu} - \int_\epsilon^\infty (C+\log u)\,e^{-ixu}\frac{du}{iu}\right\} \\ &= \frac{|x|}{x}\operatorname*{Lim}_{\epsilon\to 0}\left\{\int_\epsilon^\infty (C+\log u)\,e^{i|x|u}\frac{du}{iu} - \int_\epsilon^\infty (C+\log u)\,e^{-i|x|u}\frac{du}{iu}\right\}. \end{aligned}$$

After some manipulation it follows that

$$g_1(x) = -\sqrt{\left(\frac{\pi}{2}\right)}\,\frac{|x|}{x}\log|x|. \tag{A 4}$$

In dealing with the second and third curly bracket terms in equation (A 2), which we call $g_2(x)$ and $g_3(x)$ in the physical plane, respectively, the complex ω-plane can be cut from $-\infty-i\beta^2/\lambda$ to $-i\beta^2/\lambda$ and $-\infty-i/\lambda$ to $-i/\lambda$ in each case, and the real ω-axis contour shifted down to positions just above each branch cut, indented above the singularity, and running on to $+\infty - i\beta^2/\lambda$ and $+\infty - i/\lambda$ respectively. Taking $g_2(x)$, we find that

$$\sqrt{(2\pi)}\, e^{\beta^2 x/\lambda}\, g_2(x) = \int_{-\infty}^{\infty} (C + \log z)\, e^{-ixz} \frac{dz}{-iz}, \tag{A 5}$$

where z is a new variable related to ω by

$$z = \omega + i\beta^2/\lambda. \tag{A 6}$$

It follows from (A 5) that $g_2(x) = 0$ for $x < 0$. When $x > 0$,

$$\sqrt{(2\pi)}\, e^{\beta^2 x/\lambda} g_2(x) = \int_{\epsilon}^{\infty} (C + \log \xi + i\pi)\, e^{ix\xi} \frac{d\xi}{i\xi} + \int_0^{\pi} (C + \log \epsilon\, e^{i\theta})\, e^{-ix\epsilon e^{i\theta}}\, d\theta - \int_{\epsilon}^{\infty} (C + \log \xi)\, e^{-ix\xi} \frac{d\xi}{i\xi}, \tag{A 7}$$

in the limit as $\epsilon \to 0$ (z is put equal to ξ on the real z-axis and equal to $\epsilon\, e^{i\theta}$ on the indentation around the branch point). The integrals in (A 7) can be rearranged, and it is then possible to show that

$$\left.\begin{aligned} g_2(x) &= -\sqrt{(2\pi)}\, e^{-\beta^2 x/\lambda} \log x + i\pi^2 \frac{e^{-\beta^2 x/\lambda}}{\sqrt{(2\pi)}} \quad \text{for} \quad x > 0, \\ &= 0 \quad \text{for} \quad x < 0. \end{aligned}\right\} \tag{A 8}$$

The value for $g_3(x)$ follows at once on setting β^2 in (A 8) equal to 1.

The function $f(y)$ in equation (A 3) can be identified as $a_0(y)$ in the notation of the text; whence, using the theorem expressed there together with the results derived above, it follows that

$$\begin{aligned} b_0(x) = a_0(x) \log(\tfrac{1}{2}\beta_f) &- \frac{1}{2}\frac{\partial}{\partial x}\int_{-\infty}^{x} a_0(y) \log(x-y)\, dy \\ &+ \frac{1}{2}\frac{\partial}{\partial x}\int_{x}^{\infty} a_0(y) \log(y-x)\, dy - \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\beta^2}{\lambda}\right)\int_{-\infty}^{x} a_0(y)\, e^{-\beta^2(x-y)/\lambda} \log(x-y)\, dy \\ &+ \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{\lambda}\right)\int_{-\infty}^{x} a_0(y)\, e^{-(x-y)/\lambda} \log(x-y)\, dy. \end{aligned} \tag{A 9}$$

The imaginary terms in $g_2(x)$ and $g_3(x)$ are found to cancel out in the expression for $b_0(x)$, as indeed they must. If the indicated operations are performed on the integrals in (A 9) and the value $US'(x)/2\pi$ substituted for $a_0(x)$, the result (36) is obtained. We note that $a_0(x) = 0$ for $x < 0$ and $x > L$; whence the limits in equation (36) follow.

Case (ii)

In the transonic case (see § 3, case (ii)), we write the transform of $b_0(x)$ in the same way as in (A 2) with the following differences: $|\omega|$ is replaced by ω, β^2 by $-\bar{\beta}^2$, and Z by Z'. Inversion now takes place just above the real ω-axis, indented to pass above the point $\omega = 0$. These differences have the following effects. Using a notation consistent with that of the previous case, we have

$$\left.\begin{aligned} g_1(x) &= -\sqrt{(2\pi)} \log x + \frac{i\pi^2}{\sqrt{(2\pi)}} \quad \text{for} \quad x > 0, \\ &= 0 \quad \text{for} \quad x < 0. \end{aligned}\right\} \tag{A 10}$$

Because having $-\bar{\beta}^2$ for β^2 in the g_2 term puts the relevant singularity above the real ω-axis, it follows that

$$\left.\begin{aligned} g_2(x) &= 0 \quad \text{for} \quad x > 0, \\ &= \sqrt{(2\pi)}\, e^{-\bar{\beta}^2|x|/\lambda} \log|x| \quad \text{for} \quad x < 0. \end{aligned}\right\} \tag{A 11}$$

The function $g_3(x)$ is unchanged (see (A 8) with $\beta^2 = 1$). Using the theorem (A 3), etc., it follows that

$$\begin{aligned} b_0(x) = a_0(x) \log(\tfrac{1}{2}\beta_f) - \frac{\partial}{\partial x}\int_{-\infty}^{x} a_0(y) \log(x-y)\, dy \\ + \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\bar{\beta}^2}{\lambda}\right) \int_{x}^{\infty} a_0(y)\, e^{-\bar{\beta}^2 (y-x)/\lambda} \log(y-x)\, dy \\ + \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{\lambda}\right) \int_{-\infty}^{x} a_0(y)\, e^{-(x-y)/\lambda} \log(x-y)\, dy, \end{aligned} \tag{A 12}$$

the imaginary parts of the $g(x)$ functions cancelling out as before. With a little rearrangement and the substitution of the value for $a_0(x)$, we easily obtain equation (43), provided that we let y approach x in the last two integrals in (43) like $x+\epsilon$ and $x-\epsilon$ respectively (i.e. we take a principal value).

Case (iii)

Once again we use a form of $b_0(x)$ like that in (A 2) only here we write B_f for β_f, B^2 for β^2, $i\omega$ for $|\omega|$ and Z'' for Z in the first curly bracket expression.

It follows that the $g_2(x)$ and $g_3(x)$ terms are identical with those of case (i) when B^2 is written for β^2; and, using the contour described in § 3, case (iii) for the $\log(i\omega)$ term, we find that

$$\left.\begin{aligned} g_1(x) &= -\sqrt{(2\pi)} \log x \quad \text{for} \quad x > 0, \\ &= 0 \quad \text{for} \quad x < 0. \end{aligned}\right\} \tag{A 13}$$

The result (49) now follows in the usual way. We remark here that the Laplace transform could easily be used in the fully supersonic case, with its absence of upstream influence effects. The Fourier-transform treatment has been retained since the problem can then be dealt with in a unified fashion.

Appendix B *Evaluation of the relaxation drag integral*

The relaxation drag integral can be written generally in the form

$$2\pi S_0 \mathscr{C}_{Dr} = \int_0^L S''(x) \int_0^x S''(y) \left\{\mathrm{Ei}\left(\frac{y-x}{\lambda}\right) - \mathrm{Ei}\left(\alpha^2 \frac{y-x}{\lambda}\right)\right\} dy\, dx, \tag{B 1}$$

where α^2 may be β^2, $\bar{\beta}^2$ or B^2, depending on the Mach number régime. The inner integral in (B 1) is equal to $2I_\alpha - 2S'(x)\log\alpha$ in the notation of (37) *et seq.*

In it we write

$$x - y = \lambda\sigma, \tag{B 2}$$

so this inner integral can be rewritten in the form

$$2I_\alpha - 2S'(x)\log\alpha = \lambda \int_0^{x/\lambda} S''(x-\lambda\sigma)\{\mathrm{Ei}(-\sigma) - \mathrm{Ei}(-\alpha^2\sigma)\}\, d\sigma. \tag{B 3}$$

The cross-sectional area $S(x)$ of a wide variety of body shapes can be expressed as a polynomial in x, whence it follows from equation (B 3) that we shall have to deal with a number of integrals of the type

$$I_n = \int_0^{x/\lambda} \sigma^n\{E_1(\alpha^2\sigma) - E_1(\sigma)\}\, d\sigma. \tag{B 4}$$

In writing (B 4), use has been made of the rather more convenient notation for the exponential integral, which puts $E_1(\sigma) = -\mathrm{Ei}(-\sigma)$, etc. (see Erdelyi, Magnus, Oberhettinger & Tricomi 1953). Then, using the results given in this reference, it is found that

$$(n+1)\, I_n = \Gamma(1+n)\,[\alpha^{-2(n+1)}-1] - \alpha^{-2(n+1)}\,[\Gamma(1+n, \alpha^2 x/\lambda) - (\alpha^2 x/\lambda)^{n+1}\, E_1(\alpha^2 x/\lambda)] + \Gamma(1+n, x/\lambda) - (x/\lambda)^{n+1}\, E_1(x/\lambda). \tag{B 5}$$

The functions $\Gamma(1+n, x/\lambda)$, etc., in (B 5) are one form of the incomplete gamma function

$$\Gamma(1+n,\, x/\lambda) = \int_{x/\lambda}^{\infty} e^{-t} t^n\, dt. \tag{B 6}$$

It follows that I_α can now be expressed in terms of a sum involving the appropriate I_n's with suitable coefficients. The restriction to pointed-nosed bodies ensures that the body radius $R(x)$ will behave like x, at worst, in the nose region. Hence n in (B 4) will never be less than zero.

In evaluating the drag integral in (B 1), we now encounter terms like

$$J_{n,a} = \int_0^1 x^n\{\Gamma(a, x/\lambda) - \alpha^{-2a}\,\Gamma(a, \alpha^2 x/\lambda)\}\, dx, \tag{B 7}$$

$$K_n = \int_0^1 x^n\{E_1(\alpha^2 x/\lambda) - E_1(x/\lambda)\}\, dx. \tag{B 8}$$

The body length L has been set equal to 1 here. There is no loss of generality in so doing, but all lengths are hereafter measured in terms of L as the basic dimension. In other words, λ in (B 7) and (B 8) (and subsequently) stands for the ratio of relaxation length to body length. We find that

$$(n+1)\, J_{n,a} = \Gamma(a) - \gamma(a, 1/\lambda) + \lambda^{n+1}\gamma(a+n+1, 1/\lambda) - \alpha^{-2a}\,\Gamma(a) + \alpha^{-2a}\gamma(a, \alpha^2/\lambda) - \alpha^{-2(a+n+1)}\lambda^{n+1}\gamma(a+n+1, \alpha^2/\lambda), \tag{B 9}$$

where $\gamma(a,\, 1/\lambda)$, etc., stands for the other incomplete gamma function

$$\gamma(a,\, 1/\lambda) = \int_0^{1/\lambda} e^{-t} t^{a-1}\, dt. \tag{B 10}$$

Also

$$(n+1)\, K_n = \alpha^{-2(n+1)}\,\lambda^{n+1}\gamma(n+1, \alpha^2/\lambda) + E_1(\alpha^2/\lambda) - \lambda^{n+1}\gamma(n+1, 1/\lambda) - E_1(1/\lambda). \tag{B 11}$$

Some results have been computed for the body whose meridian profile is a parabolic arc meridian, for which

$$R(x) = 4\delta(x - x^2). \tag{B 12}$$

This has

$$S''(x) = 32\pi\delta^2(1 - 6x + 6x^2),$$

whence it follows from the general results given above that

$$2(32\delta)^{-2}\mathscr{C}_{Dr} = \frac{1}{5}\left[\frac{\lambda}{\alpha^2}(1-e^{-\alpha^2/\lambda})-\lambda(1-e^{-1/\lambda})\right]$$

$$-\frac{1}{2}\left[\left(\frac{\lambda}{\alpha^2}\right)^2\{1-e^{-\alpha^2/\lambda}e_1(\alpha^2/\lambda)\}-\lambda^2\{1-e^{-1/\lambda}e_1(1/\lambda)\}\right]$$

$$+3\left[\left(\frac{\lambda}{\alpha^2}\right)^4\{1-e^{-\alpha^2/\lambda}e_3(\alpha^2/\lambda)\}-\lambda^4\{1-e^{-1/\lambda}e_3(1/\lambda)\}\right]$$

$$-24\left[\left(\frac{\lambda}{\alpha^2}\right)^6\{1-e^{-\alpha^2/\lambda}e_5(\alpha^2/\lambda)\}-\lambda^6\{1-e^{-1/\lambda}e_5(1/\lambda)\}\right], \quad \text{(B 13)}$$

provided that we take $S_0 = \pi\delta^2$.

The functions e_1, e_3 and e_5 appearing in (B 13) are the truncated exponential series

$$e_n(z) = \sum_{m=0}^{n} \frac{z^m}{m!}. \quad \text{(B 14)}$$

They arise from the fact that, when n is an integer, the incomplete gamma function $\gamma(1+n, z)$ can be expressed in the form $n![1-e^{-z}e_n(z)]$.

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